The Sixfold Path to understanding ⟨bra|ket⟩ Notation

Start by seeing the three things that ⟨bra|ket⟩ notation can represent.

Then build understanding by looking in detail at each of those three in a progressive fashion.

Add the preview of what you will see elsewhere, and you are fully prepared for tackling more advanced treatments without losing your way in their typically erratic and incomplete introductions.

Although if you already know bra-kets in some detail, do read the What You Will See Elsewhere appendix first, since this path is different and deliberately not cluttered with references or footnotes.

Historical Note

Dirac introduced bra-ket notation in a 1939 article, and included it in the 1947 third edition of his Principles of Quantum Mechanics (not the 1930 first edition of that book as is often wrongly reported). The article was published in Mathematical Proceedings of the Cambridge Philosophical Society (1939), 35 : pp 416-418, with a title of "A New Notation for Quantum Mechanics", and starts with

> *In mathematical theories the question of notation, while not of primary importance, is yet worthy of careful consideration, since a good notation can be of great value in helping the development of a theory, by making it easy to write down those quantities or combinations of quantities that are important, and difficult or impossible to write down those that are unimportant.*

A.1 States

Given a and b, two sets of possible states of a system, the bra-ket $\langle a|b\rangle$ represents the probability of finding the system in one of the a states rather than one of the non-a states within b (or vice-versa).

Typically, a will be a single state of interest (such as detection of a particle at a particular location), and b will be all the possible states (of the particle in question).

The bra-ket $\langle a|b\rangle$ is shorthand for a bra multiplied by a ket, $\langle a|\cdot|b\rangle$, where the bra and ket taken individually are both ways of representing states.

Bra and ket for a given state are related, but differ in some way that satisfies probability calculation, so $\langle a|$ and $|a\rangle$ both refer to the same state a, and $\langle a|a\rangle$ represents a probability of one.

Actual mathematical content of the bra versus the ket is deliberately not specified – Dirac introduced bra-ket notation so that principles of Quantum Mechanics could be developed without concern for the underlying mathematics.

How they are multiplied is also not specified, though the result must be something that can be reduced to a simple number for use as a probability.

A.2 Vectors

Note: This section uses non-standard notation - see the Notation for Vectors and Matrices appendix.

Given column vectors ${}^{\downarrow}a$, ${}^{\downarrow}b$, the bra-ket $\langle a|b\rangle$ represents the dot product $\vec{a}^{*}\cdot{}^{\downarrow}b$, ie the adjoint of ${}^{\downarrow}a$ multiplied by ${}^{\downarrow}b$, and is shorthand for a bra multiplied by a ket, $\langle a|\cdot|b\rangle$, where the bra and ket taken individually are both ways of representing states of a system.

Typically, a will describe a single state of interest (such as detection of a particle at a particular location), and b will be all the possible states (of the particle in question).

The bra and ket for any given state are related by each being the adjoint of the other, so

if bra $\langle a|$ were $[a_i]_N$ then ket $|a\rangle$ would be ${}^{\downarrow}[a_i^{*}]_N$,

and conversely if ket $|b\rangle$ is ${}^{\downarrow}[b_i]_N$ then bra $\langle b|$ is $[b_i^{*}]_N$,

where * is the complex conjugate.

It follows that $\langle a|a\rangle$ is a real number, and $\langle a|b\rangle$ is a complex number from which a real-number magnitude $|\langle a|b\rangle|^2$ can be derived, and these can be normalised to represent probabilities.

A.3 Integrals

Given two functions a and b over some coordinate space of N generalised dimensions q_i , the bra-ket $\langle a|b\rangle$ represents $\int a^{*}b\ dq$, where function a^* is the complex conjugate version of function a, and dq is shorthand for $dq_1 ... dq_N$, integration over all N dimension variables q_i .

Typically, a will describe a single state of interest (such as detection of a particle at a particular location), and b will be the wavefunction Ψ (for all the possible states of the particle in question).

The bra-ket $\langle a|b\rangle$ is shorthand for the above integral, while the bra and ket taken individually, $\langle a|$ and $|b\rangle$, are both functions representing system states, with bra $\langle a|$ the complex conjugate function a^* of its related state function a, and ket $|b\rangle$ simply the state function b.

Since the value of an integral is in general a complex number, a real-number magnitude $|\langle a|b\rangle|^2$ can be derived from the bra-ket, and be normalised to represent a probability.

B.1 State transformation and combination

As outlined in section A.1, bra times ket, $\langle \alpha|\beta \rangle$ or the equivalent $\langle \alpha||\beta \rangle$ or $\langle \alpha|\cdot|\beta \rangle$, is a complex number, but to be able to work with bra-kets as generic representations of system states the rules for manipulating bras $\langle \alpha|$ and kets $|\alpha \rangle$ must be defined.

When bras, kets, and complex numbers are combined in an expression, together with operators for which multiply is used to denote action (either to the left or the right), there are some restrictions:

- Addition cannot be mixed - bras cannot be added to kets, nor operators to numbers, etc
- There is no self-multiplication of bras or kets - bras cannot be multiplied by bras, nor kets by kets
- Multiplying of a bra and a ket is not commutative - bra times ket is not the same as ket times bra
- Multiplying of operators is not commutative - operation A then B is not the same as B then A
- Operators bypass numbers with no effect, and cannot act on kets to the left nor bras to the right

but otherwise the usual associative, commutative, and distributive laws of arithmetic hold, so eg $(\langle \boldsymbol{\alpha}|\boldsymbol{\Omega})\,|\beta \rangle = \langle \alpha|\,(\boldsymbol{\Omega}|\boldsymbol{\beta}\rangle)$ and $\langle \alpha|\,(\boldsymbol{\Omega}+\boldsymbol{\Theta})\,|\beta \rangle = \langle \alpha|\boldsymbol{\Omega}|\beta \rangle + \langle \alpha|\boldsymbol{\Theta}|\beta \rangle$ where Ω and Θ are operators.

All operators of interest are linear, changing bras on the left into other bras, and kets on the right into other kets, commuting with no effect on numbers either left or right, eg $\Omega c|\alpha \rangle = c\,\Omega|\alpha \rangle = c|\beta \rangle$.

The concept of a complex conjugate, denoted by * and defined as $(x+iy)^* = (x-iy)$, is generalised to an adjoint, denoted by † and defined for complex numbers, bras, kets, and N-element expressions (including operators), respectively, as

$$\begin{aligned} c^\dagger &= c^* \\ \langle \alpha|^\dagger &= |\alpha\rangle \\ |\alpha \rangle^\dagger &= \langle \alpha| \\ (e_1 \dots e_N)^\dagger &= e_N^\dagger \dots e_1^\dagger \end{aligned}$$

The definition of adjoint operators is implied by the above definition of the effect of taking the adjoint of an expression, which is to reverse the element order and replace elements by their adjoints, so

$$\begin{aligned} \langle \alpha|\,\Omega^\dagger &= (\Omega\,|\alpha\rangle)^\dagger = \langle \beta_1| \\ \Omega^\dagger|\alpha \rangle &= (\langle \alpha|\Omega)^\dagger = |\beta_2\rangle \end{aligned}$$

An operator that is its own adjoint, $\Omega^\dagger = \Omega$, is said to be self-adjoint (and then $\beta_1 = \beta_2$ above).

The inside of a bra or ket is just a label, and it is only the complete $\langle \alpha|$ or $|\alpha \rangle$ that has mathematical meaning, so multipliers and operators cannot, strictly speaking, be brought inside, but it is convenient to adopt notational shorthand that does so, for numbers:

$$\begin{aligned} \langle c\,\alpha| &= c\langle \alpha| = \langle \alpha|c \\ |c\,\alpha \rangle &= c|\alpha \rangle = |\alpha\rangle c \end{aligned}$$

and for operators:

$$\begin{aligned} \langle \alpha\,\Omega| &= \langle \alpha|\Omega \\ |\Omega\,\alpha \rangle &= \Omega|\alpha\rangle \end{aligned}$$

allowing for example $\langle \alpha|\Omega|\beta \rangle$ to be made more specific when needed as either $\langle \alpha\,\Omega|\beta \rangle$ or $\langle \alpha|\Omega\beta \rangle$.

Some useful results that follow from the rules and definitions are:

Ket times bra, $|\alpha\rangle\langle\beta|$, is an operator, with $(|\alpha\rangle\langle\beta|)\,|\varphi\rangle = c|\alpha\rangle$ and $\langle\varphi|\,(|\alpha\rangle\langle\beta|) = d\,\langle\beta|$.

Taking the adjoint of an adjoint is the identity operation, ie $((...)^{\dagger})^{\dagger} = (...)$.

Taking the adjoint of a bra-ket reverses the order, ie $\langle\alpha|\beta\rangle^{*} = \langle\beta|\alpha\rangle$.

Taking the adjoint of a ket-bra reverses the order, ie $(|\alpha\rangle\langle\beta|)^{\dagger} = |\beta\rangle\langle\alpha|$.

In a bra-ket, the bra states may be associated with a subset of the ket states or vice-versa, depending on what sets of states the symbols α and β represent, and their bra-ket can also be expressed either way, $\langle\alpha|\beta\rangle$ or $\langle\beta|\alpha\rangle$, since being complex conjugates of each other those two bra-kets have the same magnitude and therefore represent the same probability. If they are the same set of states α, then $\langle\alpha|\alpha\rangle=1$, while for two states α and β that are incompatible $\langle\alpha|\beta\rangle=0$.

If $|\alpha_1\rangle$ and $|\alpha_2\rangle$ are distinct states, then $|\beta\rangle = c_1|\alpha_1\rangle+c_2|\alpha_2\rangle$ is also a possible state, where the c_i are complex numbers.

Each bra $\langle\alpha|$ is called the dual of its corresponding ket $|\alpha\rangle$, and since they are defined as adjoints of each other, the dual correspondence for the preceding $|\beta\rangle$ is

$$\langle\beta| \iff |\beta\rangle$$
$$c_1^{*}\langle\alpha_1|+c_2^{*}\langle\alpha_2| \iff c_1|\alpha_1\rangle+c_2|\alpha_2\rangle$$

Each physical measurement of some value of a system is represented by a self-adjoint operator M that has distinct values v_i paired with distinct states s_i (eigenvalues and eigenstates) that satisfy

$$\mathrm{M}|s_i\rangle = v_i|s_i\rangle$$

for which the following are true:

$$\langle s_i|\mathrm{M} = \langle s_i|\, v_i$$
$$\Sigma_i|s_i\rangle\langle s_i| = \mathrm{I}$$

the v_i are real

the set of kets $\{\, |s_i\rangle \,\}$ is a mutually orthonormal set of basis kets

so the $|s_i\rangle$ can be used to represent the ket of any related state α as

$$|\alpha\rangle = \Sigma_i\,\langle s_i\,|\alpha\rangle\,|s_i\rangle = \Sigma_i\, c_i\,|s_i\rangle$$

where $|c_i|^2 = |\langle s_i|\alpha\rangle|^2$ is the probability of a measurement yielding a value of v_i (corresponding to s_i).

Note that time is not considered a measureable property, but instead the states and numbers evolve with time, and it is convenient to represent that within bras and kets, and for numbers, as

$$|\alpha(t)\rangle = \Sigma_i\, c_i(t)\,|s_i(t)\rangle$$

The time evolution of a system typically matches a self-adjoint operator known as the Hamiltonian, which represents the conserved total energy and has the form

$$H\,|\alpha(t)\rangle = i\hbar\frac{\partial|\alpha(t)\rangle}{\partial t}$$

. . . . *Notes on unitary and projection operators will be added here.* . . . .

Some Derivations

Derivation of many of the useful results noted above is straightforward.

Ket times bra is an operator on kets to the right since

$$(|\alpha\rangle\langle\beta|)\,|\boldsymbol{\varphi}\rangle \;=\; |\alpha\rangle\,(\langle\beta||\varphi\rangle) \;=\; |\alpha\rangle c \;=\; \boldsymbol{c}|\boldsymbol{\alpha}\rangle$$

and on bras to the left since

$$\langle\boldsymbol{\varphi}|\,(|\alpha\rangle\langle\beta|) \;=\; (\langle\varphi||\alpha\rangle)\langle\beta| \;=\; \boldsymbol{d}\,\langle\boldsymbol{\beta}|$$

while the other possible combinations are not meaningful since they would multiply bra with bra or ket with ket, $(|\alpha\rangle\langle\beta|)\langle\varphi|$ and $|\varphi\rangle(|\alpha\rangle\langle\beta|)$.

Taking the adjoint reverses the order of a bra-ket since applying the definitions of adjoint gives

$$\langle\alpha|\beta\rangle^\dagger = (\langle\alpha||\beta\rangle)^\dagger = |\beta\rangle^\dagger\langle\alpha|^\dagger = \langle\beta||\alpha\rangle = \langle\beta|\alpha\rangle$$

and for a complex number † and * are the same, so $\langle\alpha|\beta\rangle^* = \langle\beta|\alpha\rangle$; and similarly for a ket-bra

$$(|\alpha\rangle\langle\beta|)^\dagger = \langle\beta|^\dagger\,|\alpha\rangle^\dagger = |\beta\rangle\langle\alpha|$$

For the states of a measurement operator, $\Sigma_i|s_i\rangle\langle s_i| = \mathrm{I}$, the identity operator, since

$$|\alpha\rangle = \Sigma_i c_i|s_i\rangle = \Sigma_i\,|s_i\rangle c_i = \Sigma_i\,|s_i\rangle\langle s_i|\alpha\rangle = (\Sigma_i\,|s_i\rangle\langle s_i|)\,|\alpha\rangle\,.$$

Proving that a measurement operator's kets form an orthonormal basis set is beyond this introduction, but showing that the values are real, $v_i \in \mathbb{R}$, and the states are orthogonal, $\langle s_i|s_j\rangle = 0$ if $i \neq j$, is simple, along with operation on a bra.

One way is to take the adjoint of the definition for $|s_i\rangle$ giving $\langle s_i|\mathrm{M}^\dagger = v_i^*\langle s_i|$, then multiply that on the right by $|s_j\rangle$, and also multiply the definition for $|s_j\rangle$ on the left by $\langle s_i|$ to give

$$\langle s_i\mathrm{M}^\dagger|s_j\rangle = v_i^*\langle s_i|s_j\rangle \text{ and } \langle s_i|\mathrm{M}\,s_j\rangle = v_j\langle s_i|s_j\rangle,$$

but $\mathrm{M}^\dagger = \mathrm{M}$ so $v_i^*\langle s_i|s_j\rangle = v_j\langle s_i|s_j\rangle$,

where for $i = j$ we have $\langle s_i|s_i\rangle \neq 0$ so $v_i^* = v_i$ and therefore $v_i \in \mathbb{R}$,

while for $i \neq j$ in general $v_i^* \neq v_j$ so $\langle s_i|s_j\rangle = 0$.

Then since both $\mathrm{M}^\dagger = \mathrm{M}$ and $v_i = v_i^*$, and $\langle s_i|\mathrm{M}^\dagger = v_i^*\langle s_i|$, it follows that $\langle s_i|\mathrm{M} = \langle s_i|\,v_i$.

Another way is to start similarly, with $\mathrm{M}^\dagger = \mathrm{M}$ and the adjoint operator definition giving

$$\langle s_i|\,\mathrm{M} = \langle s_i|\,\mathrm{M}^\dagger = (\mathrm{M}\,|s_i\rangle)^\dagger = (v_i|s_i\rangle)^\dagger = \langle s_i|v_i^*$$

which applied to $\langle s_i\mathrm{M}|s_i\rangle = \langle s_i|\mathrm{M}\,s_i\rangle$ gives $\langle s_i|v_i^*|s_i\rangle = \langle s_i|v_i|s_i\rangle$

so $v_i^* = v_i$ and $\langle s_i|\mathrm{M} = \langle s_i|v_i$, and then orthogonality follows from considering two pairs

$$\mathrm{M}|s_i\rangle = v_i|s_i\rangle \text{ and } \mathrm{M}|s_j\rangle = v_j|s_j\rangle$$

where multiplying each on the left by the other's bra gives

$$\langle s_j|\mathrm{M}|s_i\rangle = \langle s_j|v_i|s_i\rangle \text{ and } \langle s_i|\mathrm{M}|s_j\rangle = \langle s_i|v_j|s_j\rangle$$

in which the values v can be commuted left to give

$$\langle s_j|\mathrm{M}|s_i\rangle = v_i\langle s_j|s_i\rangle \text{ and } \langle s_i|\mathrm{M}|s_j\rangle = v_j\langle s_i|s_j\rangle$$

then taking the adjoint of both sides of the second, and M and v_j both being self-adjoint, gives

$$\langle s_j|\mathrm{M}|s_i\rangle = v_j\langle s_j|s_i\rangle$$

and consistency with the first requires

$$\text{either } v_j = v_i \text{ or } \langle s_j|s_i\rangle = 0$$

but the v_s are distinct,

so $\langle s_j|s_i\rangle$ must vanish.

B.2 Vector and matrix arithmetic

Note: This section uses non-standard notation - see the Notation for Vectors and Matrices appendix.

As outlined in section A.2, the abstract definitions of section B.1 can be made concrete by considering a bra $\langle\alpha|$ to be a row vector $[a_i]_N$, and a ket $|\beta\rangle$ to be a column vector ${}^{\downarrow}[b_i]_N$, with each combination of N values describing a state of the system, to which operators in the form of square matrices of dimension N can be applied.

The abstract arithmetic restrictions of section B.1 are then the familiar rules:

- Addition cannot be mixed
  - row vectors cannot be added to column vectors, nor matrices to numbers, etc
- There is no self-multiplication of row or column vectors
  - row vectors cannot be multiplied by row vectors, nor column by column
- Multiplying of row and column vectors is not commutative
  - row vector times column vector is not the same as column vector times row vector
- Multiplying of matrices is not commutative
  - matrix A times matrix B is not the same as matrix B times matrix A
- Matrices bypass numbers with no effect,
  and cannot be multiplied by column vectors to the left nor row vectors to the right

and writing × for multiplication regardless of what kind it is (since that is determined by the order of the operands), the usual rules of multiplication correspond to their abstract bra-ket and operator equivalents from section B.1 as follows:

$$\begin{array}{llll}
[[a_{ij}]]_N \times [[b_{ij}]]_N & = [[\Sigma_k a_{ik} b_{kj}]]_N & \qquad & \Omega\Theta = \Phi \\
[[a_{ij}]]_N \times {}^{\downarrow}[b_j]_N & = {}^{\downarrow}[\Sigma_k a_{ik} b_k]_N & & \Omega|\alpha\rangle = |\beta\rangle \\
[a_i]_N \times [[b_{ij}]]_N & = [\Sigma_k a_k b_{kj}]_N & & \langle\alpha|\Omega = \langle\beta| \\
[a_i]_N \times {}^{\downarrow}[b_i]_N & = \Sigma_k a_k b_k & & \langle\alpha|\beta\rangle = c \\
{}^{\downarrow}[a_i]_N \times [b_j]_N & = [[a_i b_j]]_N & & |\alpha\rangle\langle\beta| = \Omega
\end{array}$$

For a general M x N matrix of complex numbers, the adjoint is defined as interchanging rows with columns and replacing values with their complex conjugates, ie

$$[[a_{ij}]]^{\dagger}_{M,N} = [[a^*_{ji}]]_{N,M}$$

from which the adjoints for vectors follow immediately and match the bras and kets of section B.1

$$\begin{array}{lll}
[a_i]^{\dagger}_N = {}^{\downarrow}[a^*_i]_N & \qquad & \langle\alpha|^{\dagger} = |\alpha\rangle \\
{}^{\downarrow}[a_i]^{\dagger}_N = [a^*_i]_N & & |\alpha\rangle^{\dagger} = \langle\alpha|
\end{array}$$

and the matrix adjoint satisfies the operator definitions of B.1 since

$$[[\omega_{ij}]]^{\dagger}_N = [[\omega^*_{ji}]]_N$$

means

$$\begin{array}{ll}
\langle\alpha|\,\Omega^{\dagger} \equiv [a_h]_N \times [[\omega^*_{ji\,(ij)}]]_N = [\Sigma_k a_k \omega^*_{ki\,(ik)}]_N = [\Sigma_k a_k \omega^*_{ik}]_N & \qquad \langle\alpha|\,\Omega^{\dagger} = (\Omega\,|\alpha\rangle)^{\dagger} \\
(\Omega\,|\alpha\rangle)^{\dagger} \equiv (\,[[\omega_{ij}]]_N \times {}^{\downarrow}[a^*_h]_N\,)^{\dagger} = (\,{}^{\downarrow}[\Sigma_k \omega_{ik} a^*_k]_N\,)^{\dagger} = [\Sigma_k \omega^*_{ik} a_k]_N &
\end{array}$$

and

$$\begin{array}{ll}
\Omega^{\dagger}|\alpha\rangle \equiv [[\omega^*_{ji\,(ij)}]]_N \times {}^{\downarrow}[a^*_h]_N = {}^{\downarrow}[\Sigma_k \omega^*_{jk\,(kj)} a^*_k]_N = {}^{\downarrow}[\Sigma_k \omega^*_{kj} a^*_k]_N & \qquad \Omega^{\dagger}|\alpha\rangle = (\langle\alpha|\Omega)^{\dagger} \\
(\langle\alpha|\Omega)^{\dagger} \equiv (\,[a_h]_N \times [[\omega_{ij}]]_N\,)^{\dagger} = (\,[\Sigma_k a_k \omega_{kj}]_N\,)^{\dagger} = {}^{\downarrow}[\Sigma_k a^*_k \omega^*_{kj}]_N &
\end{array}$$

The expression rule and other details from section B.1 have similar equivalents.

B.3 Integration of wavefunctions

For some expressions involving wavefunctions ψ and ϕ, their complex conjugate functions ψ^* and ϕ^*, and an operator Ω , the corresponding abstract bra-ket notations are:

ψ	$\|\Psi\rangle$
ψ^*	$\langle\Psi\|$
$\int \psi^* \phi \, dq$	$\langle\Psi\|\Phi\rangle$
$\int \psi^* (\Omega\phi) \, dq$	$\langle\Psi\|\Omega\Phi\rangle$
$\int (\Omega\psi)^* \phi \, dq$	$\langle\Psi\,\Omega\|\Phi\rangle$

where dq is shorthand for $dq_1 \ldots dq_N$, integration over all values of all N variables q_i of ψ and ϕ.

While the symbols ψ and ϕ are often used to mean specifically the position and momentum wavefunctions respectively, the relationships above hold more generally.

. . . *Point by point equivalence of the section B.1 abstract* . . .
. . . *arithmetic rules (eg that kets cannot be multiplied by* . . .
. . . *kets), adjoint operator detail, and complex expressions,* . . .
. . . *as done for vectors and matrices in section B.2, is more* . . .
. . . *elusive for wavefunctions and integrals - if you know of* . . .
. . . *any web references with detailed treatments that might* . . .
. . . *be a useful basis for summarising, maalpu would like to* . . .
. . . *hear from you at http://www.maalpu.org.* . . .

Appendix I - What You Will See Elsewhere

This document ignores many mathematical details and assumptions and limitations, aiming to quickly reach a useful level, enough to allow simple applications to be worked through and to establish an understanding of context within which more complete and rigorous treatments can then be approached.

Terms like "Hilbert space", "Hermitian operator", "square integrable", etc, are conspicuously absent, because they are not relevant to basic use of bra-ket notation and related probability calculations.

In particular the simpler "adjoint" is used here rather than the more correct "Hermitian conjugate", and the more consistent and descriptive "self-adjoint" is used in place of "Hermitian" as an adjective for operators etc, because the differences, such as whether certain boundary conditions are satisfied, are not relevant at the level of this document.

This document is also more methodical than most, starting with the minimal rules and definitions for expressions and adjoints, and deriving consequences from those, rather than the typical mixed approach where rules and definitions have derivations interleaved, often with incomplete rules and with some derivable consequences presented as basic assumptions.

Conventional use of symbols varies considerably in the field, so here it is again chosen to suit, for example ψ and Ψ are deliberately avoided in the non-wavefunction sections, with meaning-neutral lowercase Greek α or β used instead (or the even simpler English *a* and *b* in the overview, which does not need complex numbers for which lowercase English *c* etc is used later in the detail sections), and operators are often simple uppercase Greek or Latin like Ω or M rather than hatted Latin such as Ô.

Elsewhere, the ket-bra $|a\rangle\langle b|$ is often called an outer product, and the bra-ket $\langle a|b\rangle$ an inner product, in line with their concrete meaning in a vector interpretation.

Lone bras can be avoided by considering a bra-ket $\langle a|b\rangle$ as merely shorthand for $(|a\rangle, |b\rangle)$ or the inner product of kets $|a\rangle$ and $|b\rangle$, but this document takes the common approach of treating bras $\langle a|$ as elements in their own right, since this simplifies working with them.

The term "dual" is only mentioned here as a pairing of bras and kets, but it in fact extends to their dual isomorphic Hilbert spaces, which map to each other mathematically, complete with the operators and basis vectors etc defined in each.

Not mentioned here is the $\langle\Omega^\dagger\alpha|$ form for operators within a bra, which is notational shorthand for $(\Omega^\dagger|\alpha\rangle)^\dagger$, usually used when the definition of operators is limited to ket space, so their effect on bras must be expressed in terms of kets. This document uses the alternate approach of each operator having corresponding dual forms defined in ket and bra spaces, allowing the simpler $\langle\alpha\Omega|$.

If bra-kets are not normalised, then $\langle a|a\rangle \neq 1$ and probabilities are given by $|\langle a|b\rangle|^2 / (\langle a|a\rangle\,\langle b|b\rangle)$.

The term "vector" is often used with bras and kets merely to indicate that they are more complicated than scalars, and that their coefficients of expansion can be used to make up a conventional state vector, albeit one with an infinite number of complex elements.

With that sense of "vector", generic bras and kets are objects making up a linear vector space whose scalars are the complex numbers.

In this document, the use of eg and etc and ie without dots (instead of e.g. and etc. and i.e.) is a personal quirk of the author, as is the use of simple quotes " (instead of left “ and right ”).

Appendix II - Some Mathematical Terms

Symbols marked *(non-standard)* are used in this document but are not standard in the field.

¯ (bar) - (uncommon in QM) complex conjugate, or (uncommon though used by Dirac) adjoint

* (star) - complex conjugate, or (less common) adjoint or Hermitian conjugate

† (dagger) - adjoint or Hermitian conjugate

^ (hat) - sometimes used to distinguish operators or matrices from vectors and scalars

$\rightarrow$ - row vector

$\downarrow$ - *(non-standard)* column vector *(see Notation for Vectors and Matrices appendix)*

$\langle ...|$ - bra

$\langle ...|...|...\rangle$ - bra-ket with operator in the middle

$\langle ...|...\rangle$ - bra-ket

$|...\rangle$ - ket

$|...|$ - magnitude (expressible for $c = x+iy$ as $\sqrt{x^2+y^2}$ or $\sqrt{c\,c^*}$)

[...] - vector or matrix, or (if two comma-separated values) commutator

[[...]] - *(non-standard)* matrix *(see Notation for Vectors and Matrices appendix)*

⟦...⟧ - *(non-standard)* matrix *(see Notation for Vectors and Matrices appendix)*

δ - see Dirac delta function

δ - see Kronecker delta

^{H} - see Hermitian conjugate

^{T} - see Transpose

adjoint - an extension of the concept of conjugate beyond complex numbers,
consistent with some form of multiplication resulting in a real number or similar,
eg the dot product of a complex vector and its adjoint is real,
and the product of a complex matrix and its adjoint is positive semi-definite

adjoint matrix - the conjugate transpose

anti-linear operator - an operator Ω which when operating within some space on elements E and F multiplied by constants c_1 and c_2 always satisfies $\Omega\,(c_1 E + c_2 F) = c_1^* \Omega\, E + c_2^* \Omega\, F$

bar - see ¯ (at the start)

column vector - set of values arranged in a column, behaving as a one-column matrix

commutator - [A,B] of two operators A,B is AB-BA, vanishing if A and B commute

complex - having real and imaginary parts, eg $x+iy$

complex conjugate - the conjugate of a complex number $x+iy$, written as $(x+iy)^* = (x-iy)$

conjugate - the sign-reversed version of a number composed of two added or subtracted elements, examples of such conjugate pairs for simple, irrational, and complex numbers, and for quaternions, respectively, being $x \pm y$, $x \pm \sqrt{y}$, $x \pm iy$, $a \pm (ib + jc + kd)$

conjugate matrix - a matrix with each element replaced by its conjugate

conjugate transpose - a transpose matrix with each element also replaced by its conjugate, including as special cases a row vector becoming a column vector and vice-versa cross product - of two real vectors $\vec{u}\ \vec{v}$ with angle between them of θ is a third vector perpendicular to the first two and with magnitude $\vec{u}\cdot\vec{v}\cdot\sin(\theta)$ based on their dot product

dagger - see † (at the start)

Dirac delta function - δ is a function with a single infinitely narrow spike that contains an area of one around the origin, so it satisfies $\int_{a-\epsilon}^{a+\epsilon} s(x)\delta(x-a)\,dx = s(a)$ for all smooth functions $s(x)$, any a, and any $\epsilon>0$ including both arbitrarily small ϵ and $\epsilon=\infty$

dot product - of two real vectors $\vec{u}\ \vec{v}$ is a scalar value, the sum of their corresponding components multiplied, ie $\Sigma_i u_i v_i$ if their components are u_i and v_i

eigenstate - a system state described by an eigenvector or equivalent

eigenvalue - the scalar value associated with an eigenvector

eigenvector - a vector E that, together with an associated scalar value e called its eigenvalue, is related to a given matrix Ω by $\Omega E = e E$

hat - see ^ (at start)

Hermitian operator - effectively another name for self-adjoint operator, differing in some assumptions when in infinite-dimensional spaces

Hermitian conjugate - effectively another name for adjoint

Hilbert space - vector space with finite, or non-denumerable infinite, dimension, here containing kets that correspond to physical states

inner product - of two vectors $\vec{u}\ \vec{v}$ is the dot product of one with the adjoint of the other, ie $\vec{u}\cdot\vec{v}^*$

Kronecker delta - δ_{mn} is equal to unity for $m = n$, and zero for $m \neq n$

linear operator - an operator Ω which when operating within some space on elements E and F multiplied by constants c_1 and c_2 always satisfies $\Omega(c_1 E + c_2 F) = c_1 \Omega E + c_2 \Omega F$

matrix - a set of values presented in rows and columns, here in practice always either square, or a single row or column (vector)

normalise - to divide by some scaling factor to reduce values to some norm, usually one

orthogonal - vectors whose dot product is zero

orthonormal - vectors that are orthogonal and unit length one (ie normalised)

outer product - of two vectors $\vec{u}\ \vec{v}$ is the matrix formed by multiplying each element of one with all the conjugates of the other, ie $[[u_i v_j^*]]$ if their components are u_i and v_j

row vector - set of values arranged in a row, behaving as a one-row matrix

self-adjoint - something that is its own adjoint

square integrable - function has a finite integral across all values for its square, ie $\int_{-\infty}^{+\infty} |f(x)|^2\,dx < \infty$

star - see * (at the start)

transpose - of a matrix has its rows and columns interchanged (and therefore a row vector becomes a column vector and vice-versa)

vector - a set of values, which may be considered a matrix with a single row or column, or a set of coordinates in an abstract space

Appendix III - Notation for Vectors and Matrices

This section describes the non-standard compact notation used in this document.

Standard notation for column vectors and matrices requires multiple lines, when in most cases all the relevant detail can be more compactly represented on one line, and even row vectors can be simplified.

This is done by using a superscript down arrow to mark a column vector, two square brackets for a matrix, and a subscript outside the brackets for the dimension (or two for non-square matrices):

> $^{\downarrow}$ marks a column vector, eg $^{\downarrow}r$, $^{\downarrow}[...]$, $^{\downarrow}r^{*} = ^{\downarrow}[...^{*}]$.
> $[x_i]_N$ is a row vector with dimension N and representative i-th element x_i .
> $^{\downarrow}[x_i]_N$ is a column vector with dimension N and representative i-th element x_i .
> $[[x_{ij}]]_N$ is a square matrix with dimension N and representative i-th row, j-th column element x_{ij} .
> $[[x_{ij}]]_{M,N}$ is an M-row by N-column matrix with representative i-th row, j-th column element x_{ij} .
> $[[a_{ji(ij)}]]_N = [[a_{ij}]]_N^{\mathrm{T}}$ includes in parentheses the original subscripts after a transpose

with the parenthetical original subscripts used when the two meanings that subscripts in general carry are different (the position in the current matrix versus a label identifying the original value).

When diagonal and off-diagonal matrix elements must be separately shown, a two-line form is needed:

> $\begin{bmatrix} d_{11} & u_{ij} \\ l_{ij} & d_{NN} \end{bmatrix}_N$ is a square matrix with dimension N and representive upper right element u_{ij} (i<j) and lower left element l_{ij} (i>j), and first and last diagonal elements d_{11} and d_{NN}.
>
> $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_N$ is a unit matrix of dimension N.

Two separate square brackets are used for the one-line case, but the narrower double square bracket for the two-line, because the one-line needs more emphasis, while the two-line notation is halfway to being the standard single bracket which is still used for a fully laid out matrix.

As well as being compact, this allows symmetries and parallels in expressions to show more clearly, as in the following derivation of standard vector multiplication rules from the matrix multiplication.

> For multiplication of two general non-square matrices A and B, made up of elements a_{ij} and b_{ij}, there must be a common dimension C for the columns of the first and the rows of the second, so using letters A and B also as subscripts for the non-common dimensions specific to matrices A and B, and writing A_i for row i of A, and $^{\downarrow}B_j$ for column j of B, multiplication is defined as
>
> $$[[a_{ij}]]_{A,C} \times [[b_{ij}]]_{C,B} = [[A_i\, ^{\downarrow}B_j]]_{A,B} = [[\Sigma_k a_{ik} b_{kj}]]_{A,B}$$
>
> For A=B=1, C=N the vector scalar product results
>
> $$[a_i]_N \times ^{\downarrow}[b_i]_N = [[A_i\, ^{\downarrow}B_j]]_{1,1} = \Sigma_i a_i b_i$$
>
> For A=B=N, C=1 the vector matrix product results
>
> $$^{\downarrow}[a_i]_N \times [b_j]_N = [[A_i\, ^{\downarrow}B_j]]_{N,N} = [[a_i b_j]]_N$$
>
> For A=1, B=C=N the matrix acts as an operator on a row vector to the left
>
> $$[a_i]_N \times [[b_{ij}]]_N = [[A_i\, ^{\downarrow}B_j]]_{1,N} = [\Sigma_k a_k b_{kj}]_N$$
>
> For A=C=N, B=1 the matrix acts as an operator on a column vector to the right
>
> $$[[a_{ij}]]_N \times ^{\downarrow}[b_j]_N = [[A_i\, ^{\downarrow}B_j]]_{N,1} = ^{\downarrow}[\Sigma_k a_{ik} b_k]_N$$
>
> Other combinations are not meaningful.